$\mathbb{R}^n$

We will almost always work in $\mathbb{R}^n$, though the value of n will change a lot. The values in $\mathbb{R}^n$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \text{though sometimes like} \qquad \begin{bmatrix} x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix}.$$

On the left is a column vector from $\mathbb{R}^n$, on the right is a row vector. The difference between the two is fairly marginal: they are written differently. In nearly all cases we will use column vectors. Also, get used to the 'x with subscript' format, it is the only way to work with arbitrarily sized spaces.

This form of vector is easy to work with. You add them by simply adding together the terms that are in the same spot in the vector (first row with first row, etc), so

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

This is the official 'vector addition' for the $\mathbb{R}^n$ spaces. Next we need the 'scalar multiplication', which means you multiply the vector by a one dimensional term like a real number (we will stick to the real numbers, though there are other options). It is equally simple, taking a vector in $\mathbb{R}^n$, $\mathbf{x} \in \mathbb{R}^n$ and a real number $a \in \mathbb{R}$ we multiply them by simply multiplying every term in $\mathbf{x}$ by a like so:

$$a\,\mathbf{x} = a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{bmatrix}.$$

The two operations have several useful (essential, really) properties. Taking $\mathbf{x}$, $\mathbf{y}$, $\mathbf{w}$ in $\mathbb{R}^n$, a and $b \in \mathbb{R}$ and the zero vector $\mathbf{0}$ in $\mathbb{R}^n$ (a vector with nothing but zeros) we get

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- $\mathbf{x} + (\mathbf{y} + \mathbf{w}) = (\mathbf{x} + \mathbf{y}) + \mathbf{w}$
- $\mathbf{x} + \mathbf{0} = \mathbf{x}$
- $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$
- $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$
- $1\,\mathbf{x} = \mathbf{x}$
- $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$
- $a(b\mathbf{x}) = (ab)\mathbf{x}$

These are all fairly easy to prove using the vector addition and scalar multiplications for $\mathbb{R}^n$.

Linear Combinations

These are fairly simple, but they will be used a lot, so get comfortable with them. A linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ is anything of the form

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + \mathbf{v}_n,$$

with each a a real number. Again, very simple, but we will define several key concepts and properties using linear combinations.

The most trivial linear combination is the one that uses the standard unit vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Example Is the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$?

We need to fine values a_1 and a_2 such that

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2a_1 + a_2 \\ 3a_1 + 2a_2 \end{bmatrix}.$$

To be true, that vector based equation has to match at both coordinates. The first coordinate value gives us the equation $-1 = 2a_1 + a_2$, which can be easily converted into $a_2 = -1 - 2a_1$. The second is

$$1 = 3a_1 + 2a_2 \quad \Longrightarrow \quad 1 = 3a_1 - 2 - 4a_1 \quad \Longrightarrow \quad 3 = -a_1$$

leading to $a_1 = -3$ and $a_2 = 5$. The final answer is, basically, 'yes', though it is good to include

$$-3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 + 5 \\ -9 + 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Dot Products

A dot product between two vectors $\mathbf{x}$ and $\mathbf{y}$ is written simply as $\mathbf{x} \cdot \mathbf{y}$ and calculated like so:

$$\mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_n y_n.$$

You multiply the equivalent terms in $\mathbf{x}$ and $\mathbf{y}$ then add them all up, resulting in a real number out of two vectors.

If we take the vector $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$ we can calculate

$$\mathbf{v} \cdot \mathbf{v} = 4 \times 4 + 3 \times 3 = 16 + 9 = 25.$$

Why mention this? Well, consider a more geometric interpretation of $\mathbf{v}$. It is, in actual fact, the hypotenuse of a triangle with remaining lengths 4 and 3. By the Pythagorean theorem, that makes the length of $\mathbf{v}$ equal to the square root of

$$4^2 + 3^2 = 16 + 9 = 25,$$

the exact same calculation as $\mathbf{v} \cdot \mathbf{v}$. This is not a coincidence, this is actually how we calculate the 'norm' of vectors, written $\|\mathbf{x}\|$. Norms are basically calculations of the size of vectors, so length, in this case.

$$\text{Norm of } \mathbf{x}, \text{ written } \|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$$

which works out nicely since $\mathbf{x} \cdot \mathbf{x}$ is always non-negative.

There are a few helpful properties of the dot product.

- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
- $(a\mathbf{x}) \cdot \mathbf{y} = a\,(\mathbf{x} \cdot \mathbf{y})$
- $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{w} = \mathbf{x} \cdot \mathbf{w} + \mathbf{y} \cdot \mathbf{w}$
- $\mathbf{x} \cdot \mathbf{0} = 0$

These are fairly easy to confirm using the definition.

This is how the dot product is calculated, how it is defined, etc. Now for an attempt to discuss what it means. The dot product $\mathbf{x} \cdot \mathbf{y}$ can be viewed as a multiplication of three components: the norm of $\mathbf{x}$, the norm of $\mathbf{y}$, and a third component that measures the difference between the directions of each vector. To be completely accurate:

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\| \cos(\theta)$$

where θ is the angle between the two vectors. Recall that $\cos(0) = 1$, so this arrangement leads quickly to

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|\,\|\mathbf{x}\|\ \cos(0) = \|\mathbf{x}\|^2 .$$

Also, recall that $\cos(90) = 0$, so if the two vectors are at 90 degrees, right angles to each other (perpendicular), then their dot product is zero.

Example: Calculate the following dot products:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad , \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix} \quad , \quad \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

It may be instructive to sketch the first $\mathbb{R}^2$ vector against the other $\mathbb{R}^2$ vectors.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2 + 2 = 0 \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 + 2 = 4$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \end{bmatrix} = -2 - 6 = -8 \qquad \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix} = 4 + 0 + 0 - 2 = 2$$

Orthogonality: if two vectors $\mathbf{x}$ and $\mathbf{y}$ have dot product $\mathbf{x} \cdot \mathbf{y} = 0$ then they are orthogonal.

Orthogonal is a more general descriptive term than perpendicular. Orthogonality applies in more exotic spaces where the vectors cannot be described as having a direction.

Projections

Take two vectors $\mathbf{x}$ and $\mathbf{y}$. The 'Projection' of $\mathbf{x}$ onto $\mathbf{y}$ is the component of $\mathbf{x}$ that has the same *direction* as $\mathbf{y}$. Honestly, visuals are the most useful here, see any text on the subject.

There is a simple dot product based way to calculate it:

$$Proj_y(x) = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}} \mathbf{y}.$$

This expression is not actually that hard to remember.

- The dot product is necessary to calculate how close $\mathbf{x}$ and $\mathbf{y}$ are in terms of direction.
- The whole thing is multiplied by $\mathbf{y}$ since this is the projection of $\mathbf{x}$ onto $\mathbf{y}$, and so has to be in the direction of $\mathbf{y}$.
- Then we get to the division by $\mathbf{y} \cdot \mathbf{y}$. This is necessary because the projection should not, in any way, depend on the size of $\mathbf{y}$, only its direction. On top of the expression $\mathbf{y}$ turns up twice, so we have its size multiplied into the expression twice. We need to divide by the size of $\mathbf{y}$ twice to cancel it out.

Example: Calculate $Proj_{\mathbf{v}_1}(\mathbf{v}_2)$, $Proj_{\mathbf{v}_2}(\mathbf{v}_1)$ and $Proj_{\mathbf{v}_3}(\mathbf{v}_4)$ using

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -3 \end{bmatrix}$$

$$Proj_{\mathbf{v}_1}(\mathbf{v}_2) = \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \frac{-1}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{13} \\ -\frac{3}{13} \end{bmatrix}.$$

$$Proj_{\mathbf{v}_2}(\mathbf{v}_1) = \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{-1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Cross Products

These are odd. They are, in many ways, the opposite of dot products. Dot products take two vectors in $\mathbb{R}^n$, for any n, and output a real value. The cross product we will use takes two vectors in $\mathbb{R}^3$ and outputs a vectors also in $\mathbb{R}^3$. Dot products are zero if the vectors are perpendicular, cross products are zero if the vectors are parallel (or in opposite directions). The definition of the dot product is simple, easy to remember, that of the cross product is not.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2y_3 - x_3y_2 \\ -(x_1y_3 - x_3y_1) \\ x_1y_2 - x_2y_1 \end{bmatrix}.$$

It can also be written

$$det \begin{bmatrix} \mathbf{i} & x_1 & y_2 \\ \mathbf{j} & x_2 & y_2 \\ \mathbf{k} & x_3 & y_3 \end{bmatrix} \qquad \text{using} \qquad \mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for those familiar with determinants.

Example

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (2)(0) - (1)(0) \\ -(1)(0) + (-2)(0) \\ (1)(1) - (2)(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}$$

The cross product is clearly orthogonal to the original two vectors.

$$\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} (-1)(-2) - (-2)(2) \\ -(2)(-2) + (-2)(3) \\ (2)(2) - (3)(-1) \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 7 \end{bmatrix}.$$

Use the dot product to check that the result is orthogonal to the originals.